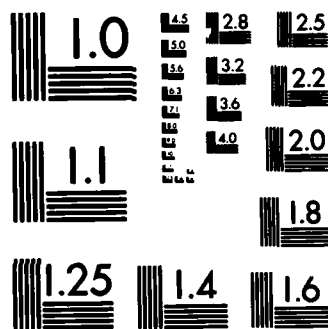


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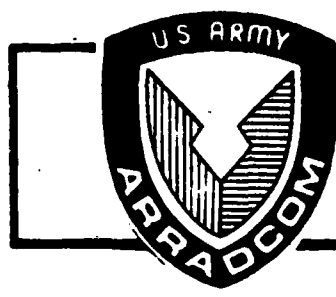
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TECHNICAL REPORT ARLCB-TR-82031

ASYMPTOTIC AND NUMERICAL METHODS FOR VECTOR SYSTEMS  
OF SINGULARLY-PERTURBED BOUNDARY VALUE PROBLEMS

Joseph E. Flaherty  
Robert E. O'Malley, Jr.

October 1982



US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND  
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## 1. INTRODUCTION

We consider singularly-perturbed two-point boundary value problems for nonlinear vector systems of the form

$$\dot{x} = f(x, y, t, \epsilon) \quad , \quad \epsilon \dot{y} = g(x, y, t, \epsilon) \quad , \quad 0 < t < 1 \quad (1a, b)$$

$$a(x(0), y(0), \epsilon) = 0 \quad , \quad b(x(1), y(1), \epsilon) = 0 \quad (1c, d)$$

where  $x$ ,  $y$ ,  $a$ , and  $b$  are vectors of dimension  $m$ ,  $n$ ,  $q$ , and  $r = m + n - q$ , respectively. We seek to find limiting solutions of problem (1) as the small positive parameter  $\epsilon$  tends to zero; however, to do this in complete generality is very difficult and beyond the grasp of our current understanding. Thus, we simplify problem (1) considerably by assuming, in addition to natural smoothness hypotheses, that (i)  $g$ ,  $a$ , and  $b$  are linear functions of the fast variable  $y$ , i.e.

$$g(x, y, t, \epsilon) = g_1(x, t, \epsilon) + G_2(x, t, \epsilon)y \quad (2a)$$

$$a(x(0), y(0), \epsilon) = a_1(x(0), \epsilon) + A_2(x(0), \epsilon)y(0) \quad (2b)$$

$$b(x(1), y(1), \epsilon) = b_1(x(1), \epsilon) + B_2(x(1), \epsilon)y(1) \quad (2c)$$

(ii) that  $G_2(x, t, \epsilon)$  has a hyperbolic splitting with  $k > 0$  stable eigenvalues and  $n - k > 0$  unstable eigenvalues for all  $x$  and  $0 < t < 1$ , and (iii) that  $q > k$  and  $r > n - k$ .

With the assumed hyperbolic splitting, we would expect  $y$  to vary rapidly relative to (the slow vector)  $x$  in narrow boundary layer regions near both  $t = 0$  and  $1$ . We thus seek limiting solutions having the form

$$x(t, \epsilon) = X(t) + O(\epsilon) \quad , \quad y(t, \epsilon) = Y(t) + \mu(\tau) + v(\sigma) + O(\epsilon) \quad (3a, b)$$

where the initial layer correction  $\mu(\tau)$  and the terminal layer correction  $v(\sigma)$ , respectively, decay to zero as the stretched variable

$$\tau = t/\epsilon \quad \text{or} \quad \sigma = (1-t)/\epsilon \quad (4a, b)$$

tend to infinity. The limiting solution  $X(t)$ ,  $Y(t)$  within  $0 < t < 1$  must necessarily satisfy the reduced system

$$\dot{X} = f(X, Y, t, 0) \quad , \quad 0 = g(X, Y, t, 0) \quad (5a, b)$$

Because  $G_2$  is everywhere nonsingular, we can use Eqs. (2a) and (5b) to determine

$$Y(t) = -G_2^{-1}(X, t, 0)g_1(X, t, 0) \quad (6)$$

in a locally unique way, and there remains the  $m$ th order nonlinear differential system (Eq. (5a)) for determining  $X(t)$ .

In order to completely specify the reduced solution we must prescribe  $m$  boundary conditions for equation (5a). We do this by providing a "cancellation law" which selects a combination of  $q-k$  initial conditions (Eq. (2b)) and of  $r - n + k$  terminal conditions (Eq. (2c)) to be satisfied by  $X$  and  $Y$ . In Section 2 we present a numerical procedure for determining the boundary conditions for the reduced system that uses an orthogonal matrix  $E(x, t)$  to reduce the matrix  $G_2(X(t), t, 0)$  to a block tridiagonal form so that the stable and unstable eigenspaces may be separated. The boundary layer corrections  $u(\tau)$  and  $v(\sigma)$  in Eqs. (3) compensate for the cancelled initial and terminal conditions, respectively, and they can be determined once  $X(t)$  has been computed (cf. Section 2). This process avoids complicated matching procedures.

In Section 3 we discuss a numerical procedure for determining the asymptotic approximation (Eq. (3)) which uses the general purpose two-point boundary value code COLSYS to solve the reduced problem and then adds numerical approximations to the boundary layer corrections. This approximation is considerably less expensive to obtain than solving the full stiff problem



numerically and it has the advantage of improving in accuracy, without any additional computational cost, as the small parameter  $\epsilon$  tends to zero. However, when  $\epsilon$  is only moderately small our asymptotic approximation may not be sufficiently accurate for some purposes, so we have developed a procedure (cf. Section 3) that generates an improved solution by using COLSYS to solve the complete problem (Eqs. (1) and (2)) with our asymptotic approximation as an initial guess. In order for this approach to succeed we must also provide COLSYS with an initial nonuniform mesh that is appropriately graded in the boundary layers (cf. Ascher and Weiss<sup>1</sup>) and we give an algorithm for constructing such a mesh in Section 3. While our procedure does not appear to be optimal, we show by an example involving the deformation of a nonlinear elastic beam (cf. Section 4) that it does offer some advantage over the more standard approach of continuation in  $\epsilon$ , where one starts with a large value of  $\epsilon$  (e.g.  $\epsilon = 1$ ) and a crude initial guess and reduces  $\epsilon$  in steps so that the mesh is gradually concentrated into boundary layer regions.

We close Section 4 with a second nonlinear beam example that is beyond the capabilities of our present methods because the matrix  $G_2$  is a function of  $y$ . Flaherty and O'Malley<sup>2</sup> analyzed this problem and showed that its solution becomes unbounded as  $\epsilon \rightarrow 0$ . We include the numerical solution of this problem in this paper in order to show one of the many challenging effects that can

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<sup>1</sup>U. Ascher and R. Weiss, "Collocation For Singular Perturbation Problems I: First Order Systems With Constant Coefficients," Technical Report 81-2, Dept. Comp. Sci., University of British Columbia, 1981.

<sup>2</sup>J. E. Flaherty and R. E. O'Malley, Jr., "Singularly-Perturbed Boundary Value Problems For Nonlinear Systems, Including a Challenging Problem For a Nonlinear Beam," Proceedings, Conference on Singulare Störungstheorie mit Anwendungen, Oberwolfach, 1981.

occur with singularly-perturbed problems.

Finally, in Section 5 we discuss our results and present some suggestions for future investigations.

## 2. ASYMPTOTIC APPROXIMATION

In order to calculate the boundary conditions for the reduced problem (Eqs. (5a) and (6)) and the boundary layer corrections  $\mu(\tau)$  and  $v(\sigma)$  we calculate the Schur decomposition of the matrix  $G_2$  at  $t = 0$  and  $t = 1$ . In particular, at  $t = 0$  we find an orthogonal matrix  $E(x(0))$  such that

$$G_2(x(0), 0, 0)E(x(0)) = E(x(0)) \begin{bmatrix} T_-(x(0)) & U(x(0)) \\ 0 & T_+(x(0)) \end{bmatrix} \quad (7)$$

where  $T_-$  is  $k \times k$  and upper triangular with the stable eigenvalues of  $G_2$ , and  $T_+$  is upper triangular with the  $n-k$  unstable eigenvalues of  $G_2$ . The decomposition (Eq. (7)) can often be obtained analytically; however, when this is not possible or practical it can be determined numerically by using the QR algorithm (cf. Golub and Wilkinson<sup>3</sup> and Ruhe<sup>4</sup> for specific procedures).

We partition  $E$  after its  $k$ th column as

$$E = [E_- \quad \bar{E}_-] \quad (8)$$

and note that  $E_-$  spans the stable eigenspace of  $G_2$  at  $t = 0$  and

$$P = E_- E_-^T \quad (9)$$

<sup>3</sup>G. H. Golub and J. H. Wilkinson, "Ill-Conditioned Eigensystems and the Computation of the Jordan Canonical Form," SIAM Review 18 (1976), pp. 578-619.

<sup>4</sup>A. Ruhe, "An Algorithm for Numerical Determination of the Structure of a General Matrix," BIT, 10 (1970), pp. 196-216.

is a projection onto this eigenspace.

Near  $t = 0$ , we assume that the terminal layer correction  $v$  is negligible, substitute the asymptotic approximation (Eq. (3)) into the differential equations (Eqs. (1a,b)), use the reduced system (Eq. (5)), and retain only the leading order terms to find that  $\mu(\tau)$  satisfies the conditionally stable system

$$d\mu/d\tau = G_2(0)\mu \quad (10)$$

where (here and below) we use the argument  $t$  to denote conditions evaluated at  $x(t) = X(t)$ ,  $t$ , and  $\epsilon = 0$ , e.g.,

$$G_2(0) := G_2(X(0), 0, 0) \quad (11)$$

Integrating Eq. (10)

$$\mu(\tau) = e^{G_2(0)\tau} \mu(0) \quad (12)$$

We require that  $\mu(\tau)$  decays as  $\tau$  increases and this will be the case provided that  $\mu(0)$  is in the stable eigenspace of  $G_2(0)$ ; thus, using Eq. (9) we require

$$\mu(0) = P(0)\mu(0) = E_-(0)E_-^T(0)\mu(0) \quad (13)$$

Using Eqs. (3), (13), and (2b) in Eq. (1b) we find that the limiting initial conditions have the form

$$a_1(0) + A_2(0) [Y(0) + E_-(0)E_-^T(0)\mu(0)] = 0 \quad (14)$$

We assume that  $A_2(0)E_-(0)$  has its maximal rank  $k$  and construct a  $q \times q$  matrix

$$L^T = [L_-^T \quad \bar{L}_-^T] \quad (15a)$$

that reduces it to row echelon form, i.e.,

$$\begin{bmatrix} L_- \\ - \\ \bar{L}_- \end{bmatrix} A_2(0)E_-(0) = \begin{bmatrix} V_- \\ 0 \end{bmatrix} \quad (15b)$$

where  $V_-$  is  $k \times k$  and nonsingular. Multiplying Eq. (14) by  $L$  and using Eqs. (13) and (15) gives the initial layer jump and the  $q-k$  initial conditions for the reduced problem, respectively, as

$$\mu(0) = -E_-(0)V_-^{-1}L_-[a_1(X(0),0) + A_2(X(0),0)Y(0)] \quad (16a)$$

and

$$\phi(X(0)) := \bar{L}_-[a_1(X(0),0) + A_2(X(0),0)Y(0)] = 0. \quad (16b)$$

We find the terminal layer jump and the  $r - (n-k)$  terminal conditions for the reduced problem in an analogous fashion with the exception that we define  $E(x(1))$  such that

$$G_2(x(1),1,0)E(x(1)) = E(x(1)) \begin{bmatrix} \hat{T}_+(x(1)) & \hat{U}(x(1)) \\ 0 & \hat{T}_-(x(1)) \end{bmatrix} \quad (17)$$

which we partition after its  $(n-k)$ th column as

$$E = [E_+ \ E_-] \quad (18)$$

In parallel with Eqs. (7) and (8), the matrices  $\hat{T}_-$ ,  $\hat{T}_+$ , and  $E_+$  contain the  $k$  stable eigenvalues, the  $n-k$  unstable eigenvalues, and span the unstable eigenspace, respectively, of  $G_2$  at  $t = 1$ . Our reasons for switching the positions of the matrices containing the stable and unstable eigenvalues of  $G_2$  is that there is no simple and stable computational procedure for finding a set of vectors that span a given subspace and are not in the leading columns of an orthogonal matrix like  $E$  (cf. Golub and Wilkinson<sup>3</sup>).

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<sup>3</sup>G. H. Golub and J. H. Wilkinson, "Ill-Conditioned Eigensystems and the Computation of the Jordan Canonical Form," SIAM Review 18 (1976), pp. 578-619.

Now, following the procedure that we used for the initial layer, we find that the terminal layer correction satisfies

$$v(\sigma) = e^{G_2(1)\sigma} v(0) \quad (19)$$

In order for  $v(\sigma)$  to decay as  $\sigma$  increases we require  $v(0)$  to be in the unstable eigenspace of  $G_2(1)$ ; thus, we take

$$v(0) = Q(1)v(0) = E_+(1)E_+^T(1)v(0) \quad (20)$$

where  $Q$  is a projection onto the  $(n-k)$  dimensional unstable eigenspace of  $G_2(1)$ .

We assume that  $B_2(1)E_+(1)$  has its maximal rank  $n-k$  and find a  $r \times r$  matrix

$$R^T = [R_+^T \quad \bar{R}_+^T] \quad (21a)$$

that reduces it to the row echelon form

$$\begin{bmatrix} R_+ \\ \bar{R}_+ \end{bmatrix} B_2(1)E_+(1) = \begin{bmatrix} V_+ \\ 0 \end{bmatrix} \quad (21b)$$

where  $V_+$  is  $(n-k) \times (n-k)$  and nonsingular. Multiplying Eq. (1d) by  $R$ , using Eqs. (2c), (3), (20), and (21), and retaining only the leading order terms we find the terminal layer jump and the  $r - (n-k)$  terminal conditions for the reduced problem, respectively, as

$$v(0) = -E_+(1)V_+^{-1}R_+[b_1(X(1),0) + B_2(X(1),0)Y(1)] \quad (22a)$$

and

$$\Psi(X(1)) := \bar{R}_+[b_1(X(1),0) + B_2(X(1),0)Y(1)] = 0 \quad (22b)$$

In the interest of brevity, we have omitted several details of our construction and have not attempted to justify the asymptotic validity of our procedure. These topics will be the subject of a forthcoming paper by O'Malley and Flaherty.<sup>5</sup>

### 3. NUMERICAL PROCEDURE

Our computational procedure consists of first solving the reduced problem (cf. Eqs. (5a), (6), (16b), and (22b)) numerically and then adding any boundary layer corrections. Since the reduced problem is not stiff we can use any good code for two-point boundary value problems (cf. Childs et al.<sup>6</sup>) and we have chosen to use the collocation code COLSYS of Ascher, Christiansen, and Russell.<sup>7</sup>

Since the reduced problem is generally nonlinear and since COLSYS solves nonlinear problems using a damped Newton method we have to supply formulas for evaluating the Jacobians of  $f$ ,  $Y$ ,  $\Phi$ , and  $\Psi$  with respect to  $X$ . We do this by providing analytical formulas for these Jacobians that neglect the influence of the derivatives of  $E$ ,  $L$ ,  $R$ , and  $G_2$ . This procedure has not failed on any of our examples; however, an alternate possibility would be to approximate the Jacobians by finite differences.

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<sup>5</sup>R. E. O'Malley, Jr., and J. E. Flaherty, "Numerical Methods For Stiff Systems of Two-Point Boundary Value Problems," to appear.

<sup>6</sup>B. Childs, M. Scott, J. W. Daniel, E. Denman, and P. Nelson (Eds.), Codes For Boundary-Value Problems in Ordinary Differential Equations, May 14-17, 1978, Lecture Notes in Computer Science, No. 76, Springer-Verlag, Berlin, 1979.

<sup>7</sup>U. Ascher, I. Christiansen, and R. D. Russell, "Collocation Software For Boundary Value ODE's," ACM Trans. Math. Software, 7 (1981), pp. 209-222.

We start the Newton iteration with a uniform mesh and the default initial guess  $X^{(0)}(t)$  for  $X(t)$  that is provided by COLSYS and calculate successive approximations  $X^{(p)}(t)$  until convergence is attained. At each iteration step we calculate an approximation  $E^{(p)}(t)$  to  $E(t)$  for  $t = 0$  and  $1$  as the Schur decomposition of  $G_2(X^{(p)}(t), t, 0)$ . In the examples of Section 4 we used analytical formulas for  $E$  rather than the numerical procedures of Golub and Wilkinson<sup>3</sup> or Ruhe.<sup>4</sup> Finally,  $L^{(p)}$  and  $R^{(p)}$  are obtained using Gaussian elimination to row reduce  $A_2(X^{(p)}(0), 0)E_{-}^{(p)}(0)$  and  $B_2(X^{(p)}(1), 0)E_{+}^{(p)}(1)$ , respectively.

When the above procedure converges we calculate boundary layer corrections  $\mu(\tau)$  and  $\nu(\sigma)$ , for a given value of  $\epsilon$ , using Eqs. (12), (16a), (19), and (22a), and add these to the reduced solution in order to get the  $O(\epsilon)$  asymptotic approximation (Eq. (3)). For moderately small values of  $\epsilon$  this approximation may not provide a sufficiently accurate representation of the solution and, in this case, we use it as an initial guess to COLSYS and solve the complete problem (Eq. (1)). Unfortunately, this procedure will fail unless we also provide COLSYS with an initial nonuniform partition

$$\pi := \{0 = t_0 < t_1 < \dots < t_N = 1\} \quad (23)$$

that is appropriately graded within the boundary layers. We seek to find  $\pi$  so that the pointwise error satisfies

$$||e(t_i)|| < \delta(1 + ||u(t_i)||) \quad , \quad i = 1, 2, \dots, N-1 \quad (24)$$

---

<sup>3</sup>G. H. Golub and J. H. Wilkinson, "Ill-Conditioned Eigensystems and the Computation of the Jordan Canonical Form," SIAM Review 18 (1976), pp. 578-619.

<sup>4</sup>A. Ruhe, "An Algorithm for Numerical Determination of the Structure of a General Matrix," BIT, 10 (1970), pp. 196-216.

where  $\delta$  is a prescribed tolerance,  $u^T := [x^T, y^T]$ ,  $e$  is the difference between  $u$  and its collocation approximation, and

$$||u(t_i)|| := \max_{1 \leq j \leq m+n} |u_j(t_i)| \quad (25)$$

We have based our condition for determining  $\pi$  on a pointwise error criteria since this seemed to work better in practice than a global criteria. This is somewhat surprising since COLSYS uses a global error criteria to select a mesh.

We assume that the final partition selected by COLSYS to solve the reduced problem satisfies equation (24) outside of boundary layer regions and we seek to refine it within the boundary layers. We further assume that derivatives of  $u$  can adequately be replaced by either  $\mu(\tau)$  or  $\nu(\sigma)$  in the left or right boundary layer, respectively.

This problem was studied by Ascher and Weiss<sup>1</sup> who showed that Eq. (24) could be approximately satisfied in the left boundary layer by choosing subinterval lengths as

$$t_i - t_{i-1} = \left( \frac{e}{\alpha_-} \right) \left[ \frac{\delta(1 + ||u(t_{i-1})||) 1/2k}{c ||\mu(t_{i-1})||} \right] \quad (26)$$

for collocation at the image of  $k$  Gauss-Legendre points per subinterval. Here  $c$  is a numerical constant and  $\alpha_-$  is the magnitude of the largest diagonal element of  $T_-(X(0))$ . A similar formula can be obtained for selecting subinterval lengths in the right boundary layer.

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<sup>1</sup>U. Ascher and R. Weiss, "Collocation For Singular Perturbation Problems I: First Order Systems With Constant Coefficients," Technical Report 81-2, Dept. Comp. Sci., University of British Columbia, 1981.



Starting with  $i = 1$  we use Eq. (26) to generate a partition until we either reach  $t = 1/2$  or a point where a subinterval length selected by Eq. (26) is larger than that used by COLSYS to solve the reduced problem. We then repeat the procedure in the right boundary layer.

We have written a computer code called SPCOL that implements the algorithms that are described in this section; thus, it (i) uses COLSYS to solve the reduced problem, (ii) calculates and adds appropriate boundary layer corrections to the reduced problem, and (iii) (optionally) suggests a mesh that can be used by COLSYS to solve the complete problem.

#### 4. EXAMPLES

In order to appraise the performance of SPCOL we have applied it to several examples involving the deformation of a nonlinear elastic beam which is resting on a nonlinear elastic foundation and is subjected to the combined action of a horizontal end thrust  $P$  and a lateral load  $p(x,t)$  per unit length (cf. Figure 1). This problem is discussed and analyzed in detail in Flaherty and O'Malley<sup>2</sup> and herein we only present the governing equations, which in dimensionless form are

$$\dot{x}_1 = \cos x_3, \quad \dot{x}_2 = \sin x_3, \quad \dot{x}_3 = y_1 \quad (27a,b,c)$$

$$\epsilon \dot{y}_1 = -y_2, \quad \epsilon \dot{y}_2 = (\lambda^2 x_2 - p) \cos x_3 - T y_1, \quad (27d,e)$$

where

$$T = \sec x_3 + \epsilon y_2 \tan x_3 \quad (27f)$$

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<sup>2</sup>J. E. Flaherty and R. E. O'Malley, Jr., "Singularly-Perturbed Boundary Value Problems for Nonlinear Systems, Including a Challenging Problem For a Nonlinear Beam," Proceedings, Conference on Singulare Störungstheorie mit Anwendungen, Oberwolfach, 1981.

The slow variables  $(x_1, x_2)$  and  $x_3$  represent the Cartesian coordinates and the tangent angle of a material particle on the centerline of the beam that was at the Cartesian location  $(t, 0)$  in the undeformed state. The fast variables  $y_1$  and  $y_2$  are the internal bending moment and transverse shear force, respectively (cf. Figure 1). Finally, the small parameter is

$$\epsilon^2 = EI/PL^2, \quad (28)$$

where  $EI$  is the flexural rigidity and  $L$  is the length of the beam; thus, our beam is much stronger in extension than it is in bending.

This example does not precisely fit our hypotheses since the axial force  $T$  is a function of the fast variable  $y_2$  and, thus,  $G_2$  also depends on  $y$ . However, our theory and methods will still apply as long as  $y$  remains bounded and  $|x_3| < \pi/2$  as  $\epsilon \rightarrow 0$ . In order to illustrate the diverse behaviors that can occur when  $y$  either does or does not remain bounded as  $\epsilon \rightarrow 0$  we present solutions for two problems both having  $\lambda = p = 1$  and which differ only in their boundary conditions. Some additional examples are presented in Flaherty and O'Malley.<sup>2,5</sup>

In our first example we take the boundary conditions as

$$\begin{aligned} x_1(0) = 0, \quad -10x_2(0) + y_2(0) = 0, \quad -x_3(0) + 10y_1(0) = 0 \\ 10x_2(1) + y_2(1) = 0, \quad 10x_3(1) + y_1(1) = 0 \end{aligned} \quad (29)$$

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<sup>2</sup>J. E. Flaherty and R. E. O'Malley, Jr., "Singularly-Perturbed Boundary Value Problems for Nonlinear Systems, Including a Challenging Problem For a Non-linear Beam," Proceedings, Conference on Singulare Störungstheorie mit Anwendungen, Oberwolfach, 1981.

<sup>5</sup>R. E. O'Malley, Jr., and J. E. Flaherty, "Numerical Methods For Stiff Systems of Two-Point Boundary Value Problems," to appear.

These supports correspond to a beam that is almost simply supported at  $t = 0$  and almost clamped at  $t = 1$ . However, perhaps due to friction, there is some coupling between lateral and rotational effects at the supports.

As we shall see,  $y$  remains bounded in this example so our methods are applicable. The orthogonal matrix

$$E(x(0)) = (1+\alpha^2)^{-1/2} \begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix} \quad (30a)$$

where

$$\alpha^2 = \sec^2 x_3(0) \quad (30b)$$

reduces

$$G_2(x(0), 0, 0) = \begin{bmatrix} 0 & -1 \\ -\alpha^2 & 0 \end{bmatrix} \quad (31)$$

to the Schur form given by equation (7) at  $t = 0$  and  $E^T$  will reduce  $G_2(x(1), 1, 0)$  to the form given by Eq. (17) at  $t = 1$ .

We solved this problem in two ways: (1) using COLSYS to solve the complete problem (Eqs. (27) and (29)) with continuation from a large to a small value of  $\epsilon$  and (2) using our code SPCOL to compute an initial asymptotic approximation and to recommend a nonuniform mesh and using this with COLSYS to calculate an improved solution. All calculations were performed in double precision on an IBM 3033 computer, used two collocation points per subinterval, and set the error tolerance  $\delta$  (cf. Eq. (24)) at  $10^{-6}$  for slow variables and  $10^{-3}$  for fast variables.

Our results for the normalized CP times and the number of subintervals (NSUB) that are either used by COLSYS or recommended by SPCOL are shown in

Tables I and II for continuation in  $\epsilon$  and our methods, respectively. Differences between our initial asymptotic approximation and the final solution obtained by COLSYS are shown for  $x_3$  and  $y_2$  at  $t = 0$  and 1 in Table III. We see that the differences decrease like  $O(\epsilon)$  as expected. Differences that are recorded as zero are less than  $10^{-8}$ . Finally, we exhibit solutions for  $x_2$ ,  $x_3$ ,  $y_1$ , and  $y_2$  in Figure 2.

TABLE I. NONLINEAR ELASTICALLY SUPPORTED BEAM. NUMBER OF SUBINTERVALS (NSUB) AND CP TIMES USED TO SOLVE THE PROBLEM BY COLSYS WITH CONTINUATION IN  $\epsilon$ . THE TOTAL CP IS THE ACCUMULATED TIME FOR THE  $\epsilon$  SEQUENCE.

$\epsilon$	NSUB	CP	Total CP
$10^{-1}$	80	8.0	8.0
$10^{-2}$	78	9.0	17.0
$10^{-4}$	78	19.5	36.5
$10^{-6}$	156	44.5	81.0
$10^{-8}$	100	19.0	100.0

TABLE II. NONLINEAR ELASTICALLY SUPPORTED BEAM. NUMBER OF SUBINTERVALS (NSUB) AND CP TIMES TO SOLVE THE PROBLEM BY SPCOL AND OBTAIN AN IMPROVEMENT BY COLSYS. THE CP TIMES FOR SPCOL INCLUDE THE TIME TO CALCULATE THE REDUCED SOLUTION WHICH WAS 4.8 TIME UNITS. CORRECTION NO. 1 USES THE MESH THAT WAS RECOMMENDED BY SPCOL. CORRECTION NO. 2 USES A MESH THAT IS TWICE AS COARSE. THE TOTAL CP IS THE SUM OF THE TIMES FOR THE SPCOL AND COLSYS SOLUTIONS.

$\epsilon$	SPCOL		COLSYS Correction No. 1			COLSYS Correction No. 2		
	Rec. No. of NSUB	CP	NSUB	CP	Total CP	NSUB	CP	Total CP
$10^{-1}$	40	4.9	100	12.0	16.9	80	12.1	16.9
$10^{-2}$	45	4.9	90	12.0	16.9	78	8.1	12.9
$10^{-4}$	54	4.9	108	16.9	21.8	66	9.2	14.1
$10^{-8}$	55	4.9	110	17.5	22.3			Failed

TABLE III. NONLINEAR ELASTICALLY SUPPORTED BEAM. DIFFERENCES BETWEEN SPCOL AND COLSYS SOLUTIONS, WHERE  $\Delta( ) := |( )_{\text{SPCOL}} - ( )_{\text{COLSYS}}|$

$\epsilon$	$\Delta x_3(0)$	$\Delta y_2(0)$	$\Delta x_3(1)$	$\Delta y_2(1)$
$10^{-1}$	$3.3 \times 10^{-1}$	$5.1 \times 10^{-2}$	$6.8 \times 10^{-1}$	$3.6 \times 10^{-1}$
$10^{-2}$	$2.8 \times 10^{-2}$	$6.6 \times 10^{-3}$	$6.1 \times 10^{-2}$	$3.9 \times 10^{-2}$
$10^{-4}$	$2.7 \times 10^{-4}$	$6.8 \times 10^{-5}$	$6.1 \times 10^{-4}$	$3.9 \times 10^{-4}$
$10^{-8}$	0	$1.3 \times 10^{-7}$	0	0

The results reported in Tables I and II need some additional explanation. The number of subintervals and CP times used with continuation depended heavily on the  $\epsilon$  sequence that was used. The results in Table I are about the best insofar as they gave the smallest total CP time for the sequence. In addition, COLSYS relies on the difference between solutions that are computed on two different partitions in order to estimate local errors. Thus, at a minimum, COLSYS would always double our suggested mesh. This is apparent in the results listed under the heading of "COLSYS Correction No. 1" in Table II. In some sense these results are encouraging insofar as they indicate that our mesh selection strategy is doing about as well as it can, at least for  $\epsilon < 10^{-2}$ . However, it seems that fewer points should be necessary, so we tried giving COLSYS an initial mesh that consisted of every other point of our recommended mesh. This is clearly a risky strategy since collocation at the Gauss-Legendre points is known to be unstable unless the mesh is sufficiently fine in the boundary layers (cf. Ascher and Weiss<sup>1</sup>). Our results using this

<sup>1</sup>U. Ascher and R. Weiss, "Collocation For Singular Perturbation Problems I: First Order Systems With Constant Coefficients," Technical Report 81-2, Dept. Comp. Sci., University of British Columbia, 1981.

are reported under the heading of "COLSYS Correction No. 2" in Table II. Some improvement is noted for  $\epsilon > 10^{-4}$ ; however, COLSYS failed to find a solution (within our prescribed limitations) when  $\epsilon = 10^{-8}$ .

In our second example we use the boundary conditions

$$\begin{aligned} x_1(0) = 0 \quad , \quad -x_2(0) + \epsilon y_2(0) = 0 \quad , \quad -x_3(0) + \epsilon^2 y_1(0) = 0 \\ x_2(1) + \epsilon y_2(1) = 0 \quad , \quad x_3(1) + \epsilon^2 y_1(1) = 0 \end{aligned} \quad (32)$$

If  $\epsilon$  were set to zero then these boundary conditions would correspond to clamped supports at  $t = 0$  and  $1$ . Since the limiting boundary conditions only involve the slow variables and since the slow vector  $x$  cannot generally satisfy all of them as  $\epsilon \rightarrow 0$  we would expect the solution to have boundary layers in these components. This in turn will force the fast vector  $y$  to become unbounded like  $O(1/\epsilon)$  at the endpoints! Thus, this problem does not have an asymptotic expansion having the form of Eq. (3); however, an appropriate asymptotic representation of a solution has been obtained by Flaherty and O'Malley.<sup>2</sup> We shall not repeat those results here, but in order to emphasize the diverse behavior that can occur in nonlinear singularly-perturbed problems, we present solutions for  $x_2$ ,  $x_3$ ,  $\epsilon y_1$ , and  $\epsilon y_2$  in Figure 3. These solutions were computed using COLSYS with continuation in  $\epsilon$ .

## 5. DISCUSSION

We have obtained asymptotic approximations for a restricted class of nonlinear singularly-perturbed boundary value problems and have shown how to

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<sup>2</sup>J. E. Flaherty and R. E. O'Malley, Jr., "Singularly-Perturbed Boundary Value Problems For Nonlinear Systems, Including a Challenging Problem For a Non-linear Beam," Proceedings, Conference on Singulare Störungstheorie mit Anwendungen, Oberwolfach, 1981.

construct them numerically and use them to suggest a nonuniform mesh that may be used as input to a two-point boundary value code in order to calculate improved solutions. Clearly this approach offers some advantages over the more standard technique of continuation in  $\epsilon$  steps; however, the picture is far from clear and several questions still remain as to how best to use asymptotic analysis in conjunction with numerical analysis.

As we have shown in our second example of Section 4, very diverse behavior in the solution of singularly-perturbed problems can result from seemingly minor changes in boundary conditions. Some phenomena cannot easily be predicted, so perhaps a sensible course to follow is to use asymptotic and numerical methods in tandem. For example, a rough numerical solution could be obtained for several values of  $\epsilon$  which could then be used to suggest the form of an asymptotic solution. The asymptotic approximation could then be used to refine the numerical solution, and so on. It is also possible that singular perturbation theory could be used to construct special methods that are appropriate for specific problems as e.g., in Flaherty and Mathon<sup>8</sup> and Ascher and Weiss.<sup>1</sup>

Throughout our discussion we have ignored the question of uniqueness. In general, multiple solutions can be expected and they must be coped with

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<sup>1</sup>U. Ascher and R. Weiss, "Collocation For Singular Perturbation Problems I: First Order Systems With Constant Coefficients," Technical Report 81-2, Dept. Comp. Sci., University of British Columbia, 1981.

<sup>8</sup>J. E. Flaherty and W. Mathon, "Collocation With Polynomial and Tension Splines For Singularly-Perturbed Boundary Value Problems," SIAM J. Sci. Stat. Comput., 1, (1980), pp. 260-289.

numerically. In Reference 9 we showed how asymptotic methods may be used to distinguish the different solutions and to provide initial guesses for a two-point boundary value code.

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<sup>9</sup>J. E. Flaherty and R. E. O'Malley, Jr., "On the Numerical Integration of Two-Point Value Problems For Stiff Systems of Ordinary Differential Equations," Boundary and Interior Layers - Computational and Asymptotic Methods, J. J. H. Miller, Editor, Boole Press, Dublin, 1980, pp. 93-102.



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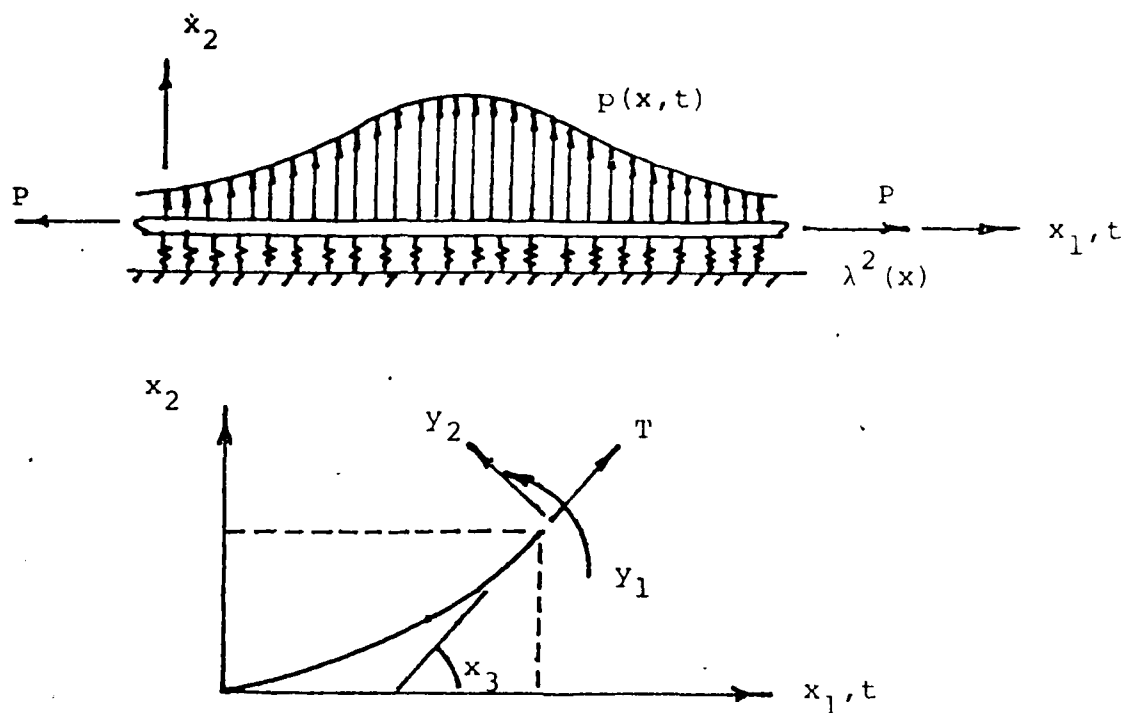


Figure 1. Geometry, loading, force, and moment conventions for nonlinear beam.

Figure 2. Numerical solution of elastically supported beam with boundary conditions given by Equations (29).

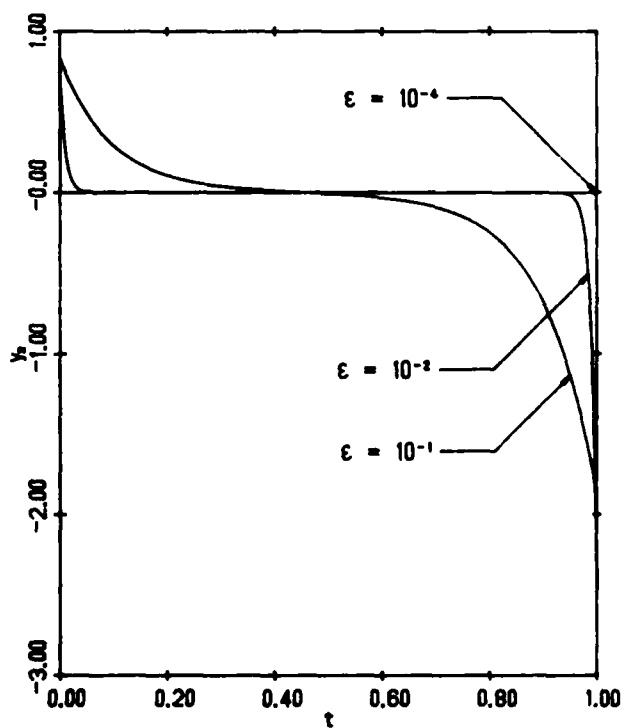
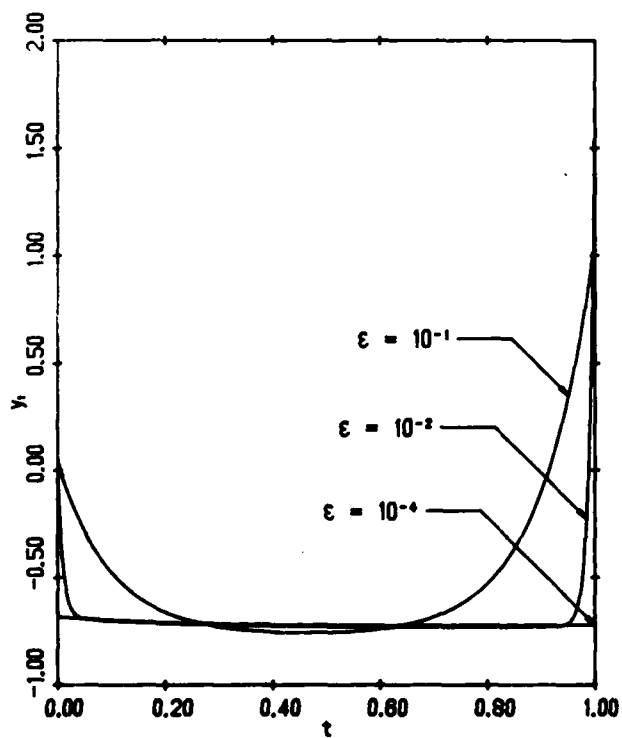
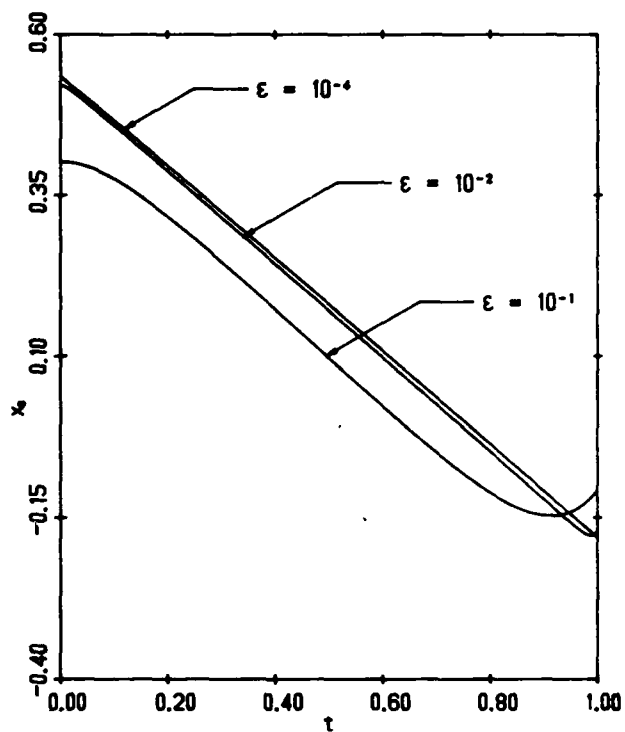
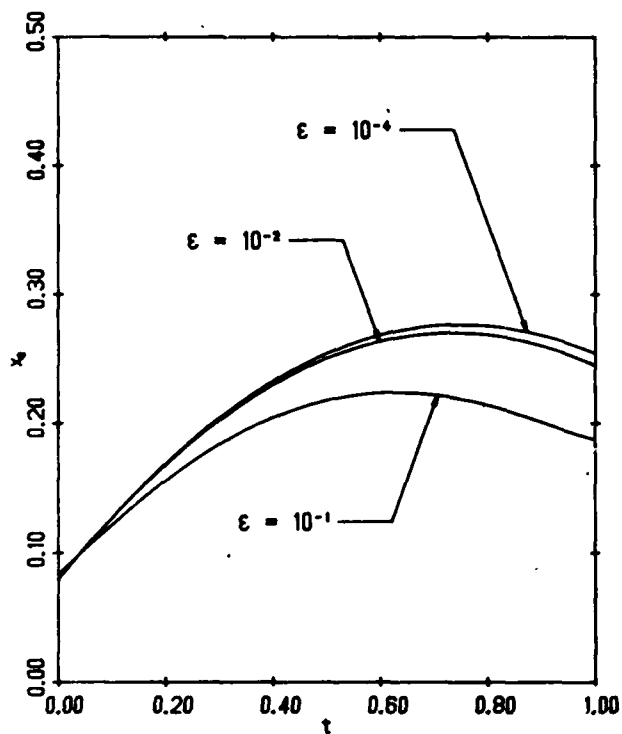
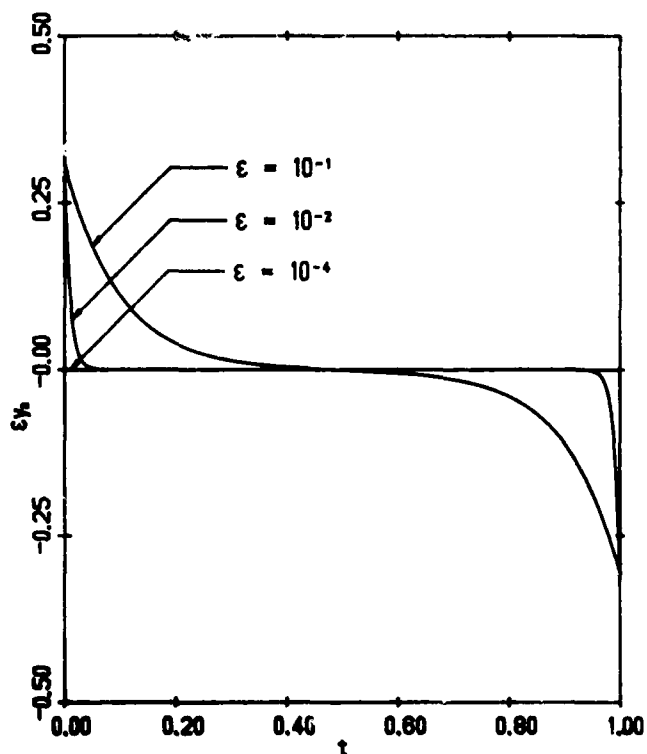
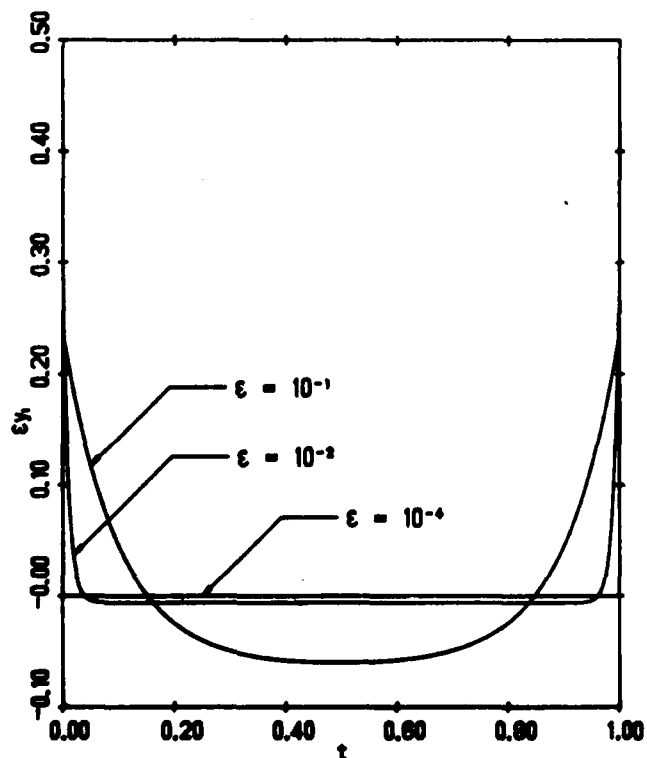
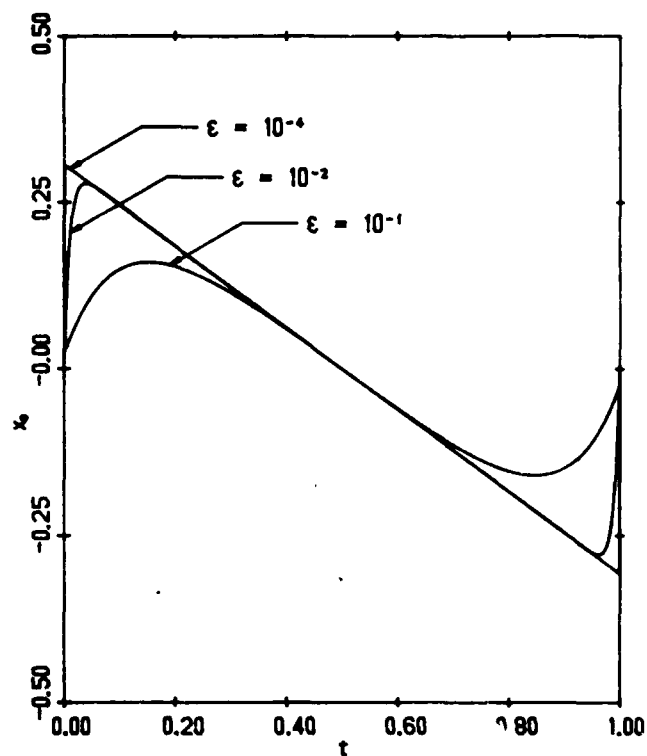
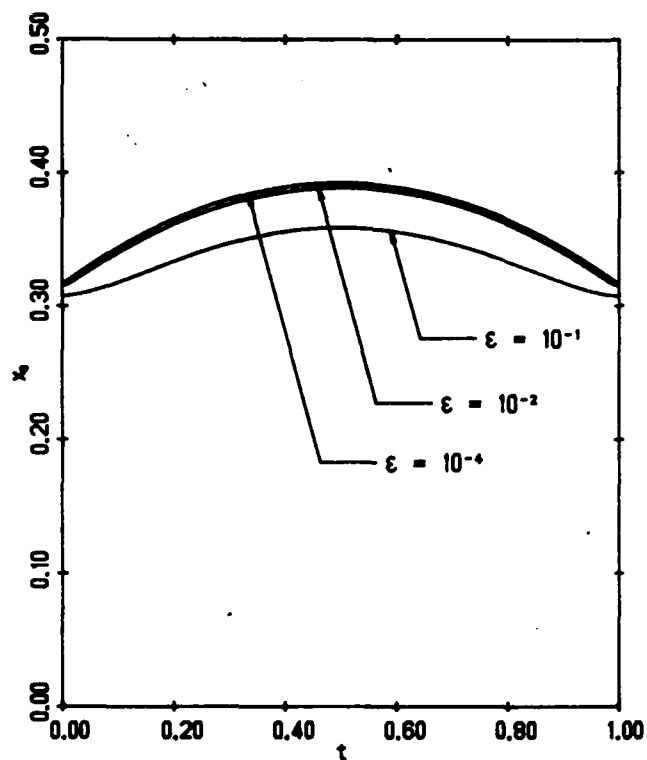


Figure 3. Numerical solution of elastically supported beam with boundary conditions given by Equations (32). Note that  $y_1$  and  $y_2$  are multiplied by  $\epsilon$ .



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